Unit F1 Further Pure Mathematics 1

IAS compulsory unit for Further Mathematics and Pure Mathematics

Externally assessed

F1.1 Unit description

Complex numbers; roots of quadratic equations; numerical solution of equations; coordinate systems; matrix algebra; transformations using matrices; series; proof.

F1.2 Assessment information

Prerequisites

A knowledge of the specification for C12, its prerequisites, preambles and associated formulae, is assumed and may be tested.

It is also necessary for students:

- to have a knowledge of location of roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval in which $f(x)$ is continuous
- to have a knowledge of rotating shapes through any angle about (0, 0)
- to be able to divide a cubic polynomial by a quadratic polynomial
- to be able to divide a quartic polynomial by a quadratic polynomial.

Examination

The examination will consist of one $1\frac{1}{2}$ hour paper. It will contain about nine questions of varying length. The mark allocations per question will be stated on the paper. All questions should be attempted.

Calculators

Students are expected to have available a calculator with at least the following keys: $+$, $-$, $\times$, $\div$, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, $\ln x$, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory. Calculators with a facility for symbolic algebra, differentiation and/or integration are not permitted.

Formulae

Formulae which students are expected to know are given overleaf and these will not appear in the booklet, *Mathematical Formulae including Statistical Formulae and Tables*, which will be provided for use with the paper. Questions will be set in SI units and other units in common usage.

This section lists formulae that students are expected to remember and that will not be included in formulae booklets.

Roots of quadratic equations

For $ax^2 + bx + c = 0$ $\quad \alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$

Series

$$\sum_{r=1}^{n} r = \frac{1}{2} n(n+1)$$

1. Complex numbers

What students need to learn:

Definition of complex numbers in the form $a + \mathrm{i}b$ and $r\cos\theta + \mathrm{i}\, r\sin\theta$.	The meaning of conjugate, modulus, argument, real part, imaginary part and equality of complex numbers should be known.
Sum, product and quotient of complex numbers.	$\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$ Knowledge of the result $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ is not required.
Geometrical representation of complex numbers in the Argand diagram.	
Geometrical representation of sums, products and quotients of complex numbers.	
Complex solutions of quadratic equations with real coefficients.	
Finding conjugate complex roots and a real root of a cubic equation with integer coefficients.	Knowledge that if z_1 is a root of $\mathrm{f}(z) = 0$ then z_1^* is also a root.

Finding conjugate complex roots and/or real roots of a quartic equation with real coefficients.	For example, (i) $f(x) = x^4 - x^3 - 5^2 + 7x + 10$ Given that $x = 2 + i$ is a root of $f(x) = 0$, use algebra to find the three other roots of $f(x) = 0$. (ii) $g(x) = x^4 - x^3 + 6x^2 + 14x - 20$ Given $g(1) = 0$ and $g(-2) = 0$, use algebra to solve $g(x) = 0$ completely.

2. Roots of quadratic equations

What students need to learn:

Sum of roots and product of roots of a quadratic equation.	For the equation $ax^2 + bx + c = 0$ whose roots are α and β then $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$.
Manipulation of expressions involving the sum of roots and product of roots.	Knowledge of the identity $\alpha^3 + \beta^3 \equiv (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$.
Forming quadratic equations with new roots.	For example, with roots $\alpha^3, \beta^3; \frac{1}{\alpha}, \frac{1}{\beta}; \frac{1}{\alpha^2}, \frac{1}{\beta^2}; \alpha + \frac{2}{\beta}, \beta + \frac{2}{\alpha};$ etc.

3. Numerical solution of equations

What students need to learn:

Equations of the form $f(x) = 0$ solved numerically by:	$f(x)$ will only involve functions used in C12.
(i) interval bisection,	
(ii) linear interpolation,	
(iii) the Newton–Raphson process.	For the Newton-Raphson process, the only differentiation required will be as defined in unit C12.

4. Coordinate systems

What students need to learn:

Cartesian equations for the parabola and rectangular hyperbola.	Students should be familiar with the equations: $y^2 = 4ax$ and $xy = c^2$
Idea of parametric equations for parabola and rectangular hyperbola.	$x = at^2, y = 2at$ and $x = ct\ , y = \frac{c}{t}$. The idea of $(at^2\ , 2at)$ and $(ct, \frac{c}{t})$ as general points on the parabola and rectangular hyperbola respectively.
The focus-directrix property of the parabola.	Concept of focus and directrix and parabola as locus of points equidistant from focus and directrix.
Tangents and normals to these curves.	Differentiation of $y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}, \qquad y = \frac{c^2}{x}$. Parametric differentiation is not required.

5. Matrix algebra

What students need to learn:

Addition and subtraction of matrices.	
Multiplication of a matrix by a scalar.	
Products of matrices.	
Evaluation of 2×2 determinants.	Singular and non-singular matrices.
Inverse of 2×2 matrices.	Use of the relation $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

6. Transformations using matrices

What students need to learn:

Linear transformations of column vectors in two dimensions and their matrix representation.	The transformation represented by $\mathbf{AB}$ is the transformation represented by $\mathbf{B}$ followed by the transformation represented by $\mathbf{A}$.
Applications of 2×2 matrices to represent geometrical transformations.	Identification and use of the matrix representation of single transformations from: reflection in coordinate axes and lines $y = \pm x$, rotation through any angle about $(0, 0)$, stretches parallel to the x-axis and y-axis, and enlargement about centre $(0, 0)$, with scale factor k, $(k \neq 0)$, where $k \in \mathbb{R}$.
Combinations of transformations.	Identification and use of the matrix representation of combined transformations.
The inverse (when it exists) of a given transformation or combination of transformations.	Idea of the determinant as an area scale factor in transformations.

7. Series

What students need to learn:

Summation of simple finite series.	Students should be able to sum series such as $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$, $\sum_{r=1}^{n} r(r^2+2)$. The method of differences is not required.

8. Proof

What students need to learn:

Proof by mathematical induction.	To include induction proofs for (i) summation of series e.g. show $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ or $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ (ii) divisibility e.g. show $3^{2n} + 11$ is divisible by 4. (iii) finding general terms in a sequence e.g. if $u_{n+1} = 3u_n + 4$ with $u_1 = 1$, prove that $u_n = 3^n - 2$. (iv) matrix products e.g. show $\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & -n \\ 9n & 3n+1 \end{pmatrix}$.

Unit F2 Further Pure Mathematics 2

IA2 optional unit for Further Mathematics and Pure Mathematics

Externally assessed

F2.1 Unit description

Inequalities; series; further complex numbers; first order differential equations; second order differential equations; Maclaurin and Taylor series; Polar coordinates.

F2.2 Assessment information

Prerequisites

A knowledge of the specifications for C12, C34 and F1, their prerequisites, preambles and associated formulae, is assumed and may be tested.

Examination

The examination will consist of one $1\frac{1}{2}$ hour paper. It will contain about eight questions of varying length. The mark allocations per question will be stated on the paper. All questions should be attempted.

Questions will be set in SI units and other units in common usage.

Calculators

Students are expected to have available a calculator with at least the following keys: $+$, $-$, $\times$, $\div$, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, $\ln x$, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory. Calculators with a facility for symbolic algebra, differentiation and/or integration are not permitted.

1. Inequalities

What students need to learn:

The manipulation and solution of algebraic inequalities and inequations, including those involving the modulus sign.	The solution of inequalities such as $\frac{1}{x-a} > \frac{x}{x-b}$, $\lvert x^2 - 1\rvert > 2(x+1)$.

2. Series

What students need to learn:

Summation of simple finite series using the method of differences.	Students should be able to sum series such as $\sum_{r=1}^{n} \frac{1}{r(r+1)}$ by using partial fractions such as $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$.

3. Further complex numbers

What students need to learn:

Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$.	Students should be familiar with $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.
De Moivre's theorem and its application to trigonometric identities and to roots of a complex number.	To include finding $\cos n\theta$ and $\sin m\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$ and also powers of $\sin\theta$ and $\cos\theta$ in terms of multiple angles. Students should be able to prove De Moivre's theorem for any integer n.
Loci and regions in the Argand diagram.	Loci such as $\lvert z-a\rvert = b$, $\lvert z-a\rvert = k\lvert z-b\rvert$, $\arg(z-a) = \beta$, $\arg\frac{z-a}{z-b} = \beta$ and regions such as $\lvert z-a\rvert \leqslant \lvert z-b\rvert$, $\lvert z-a\rvert \leqslant b$.
Elementary transformations from the z-plane to the w-plane.	Transformations such as $w = z^2$ and $w = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$, may be set.

4. First order differential equations

What students need to learn:

Further solution of first order differential equations with separable variables.	The formation of the differential equation may be required. Students will be expected to obtain particular solutions and also sketch members of the family of solution curves.
First order linear differential equations of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x.	The integrating factor $e^{\int P\,dx}$ may be quoted without proof.
Differential equations reducible to the above types by means of a given substitution.	

5. Second order differential equations

What students need to learn:

The linear second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where a, b and c are real constants and the particular integral can be found by inspection or trial.	The auxiliary equation may have real distinct, equal or complex roots. $f(x)$ will have one of the forms ke^{px}, $A + Bx$, $p + qx + cx^2$ or $m \cos \omega x + n \sin \omega x$. Students should be familiar with the terms 'complementary function' and 'particular integral'. Students should be able to solve equations of the form $\frac{d^2y}{dx^2} + 4y = \sin 2x$.
Differential equations reducible to the above types by means of a given substitution.	

6. Maclaurin and Taylor series

What students need to learn:

Third and higher order derivatives.	
Derivation and use of Maclaurin series.	The derivation of the series expansion of e^x, $\sin x$, $\cos x$, $\ln(1 + x)$ and other simple functions may be required.
Derivation and use of Taylor series.	The derivation, for example, of the expansion of $\sin x$ in ascending powers of $(x - \pi)$ up to and including the term in $(x - \pi)^3$.
Use of Taylor series method for series solutions of differential equations.	Students may, for example, be required to find the solution in powers of x as far as the term in x^4,of the differential equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, such that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$.

7. Polar coordinates

What students need to learn:

Polar coordinates (r, θ), $r \geqslant 0$.	The sketching of curves such as $\theta = \alpha$, $r = p \sec(\alpha - \theta)$, $r = a$, $r = 2a \cos\theta$, $r = k\theta$, $r = a(1 \pm \cos\theta)$, $r = a(3 + 2\cos\theta)$, $r = a\cos 2\theta$ and $r^2 = a^2 \cos 2\theta$ may be set.
Use of the formula $\frac{1}{2}\int_\alpha^\beta r^2 \, d\theta$ for area.	The ability to find tangents parallel to, or at right angles to, the initial line is expected.

Unit F3 Further Pure Mathematics 3

IA2 optional unit for Further Mathematics and Pure Mathematics — Externally assessed

F3.1 Unit description

Hyperbolic functions; further coordinate systems; differentiation; integration; vectors; further matrix algebra.

F3.2 Assessment information

Prerequisites

A knowledge of the specifications for C12, C34 and F1, their prerequisites, preambles and associated formulae, is assumed and may be tested.

Examination

The examination will consist of one $1\frac{1}{2}$ hour paper. It will contain about eight questions of varying length. The mark allocations per question will be stated on the paper. All questions should be attempted.

Questions will be set in SI units and other units in common usage.

Calculators

Students are expected to have available a calculator with at least the following keys: $+$, $-$, $\times$, $\div$, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, $\ln x$, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory. Calculators with a facility for symbolic algebra, differentiation and/or integration are not permitted.

1. Hyberbolic functions

What students need to learn:

Definition of the six hyperbolic functions in terms of exponentials. Graphs and properties of the hyperbolic functions.	For example, $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$. Students should be able to derive and use simple identities such as $\cosh^2 x - \sinh^2 x \equiv 1$ and $\cosh^2 x + \sinh^2 x \equiv \cosh 2x$ and to solve equations such as $a \cosh x + b \sinh x = c$.
Inverse hyperbolic functions, their graphs, properties and logarithmic equivalents.	E.g. $\text{arsinh}\, x = \ln[x + \sqrt{(1 + x^2)}]$. Students may be required to prove this and similar results.

2. Further coordinate systems

What students need to learn:

Cartesian and parametric equations for the ellipse and hyperbola.	Extension of work from F1. Students should be familiar with the equations: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;\ x = a\cos t,\ y = b\sin t.$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;\ x = a\sec t,\ y = b\tan t;$ $x = a\cosh t,\ y = b\sinh t.$
The focus-directrix properties of the ellipse and hyperbola, including the eccentricity.	For example, students should know that, for the ellipse, $b^2 = a^2(1 - e^2)$, the foci are $(ae, 0)$ and $(-ae, 0)$ and the equations of the directrices are $x = +\frac{a}{e}$ and $x = -\frac{a}{e}$.
Tangents and normals to these curves.	The condition for $y = mx + c$ to be a tangent to these curves is expected to be known.
Simple loci problems.	

3. Differentiation

What students need to learn:

Differentiation of hyperbolic functions and expressions involving them.	For example, $\tanh 3x$, $x\sinh^2 x$, $\frac{\cosh 2x}{\sqrt{(x+1)}}$.
Differentiation of inverse functions, including trigonometric and hyperbolic functions.	For example, $\arcsin x + x\sqrt{(1 - x^2)}$, $\frac{1}{2}\operatorname{artanh} x^2$.

4. Integration

What students need to learn:

Integration of hyperbolic functions and expressions involving them.	
Integration of inverse trigonometric and hyperbolic functions.	For example, $\int \operatorname{arsinh} x \, dx$, $\int \arctan x \, dx$.
Integration using hyperbolic and trigonometric substitutions.	To include the integrals of $1/(a^2 + x^2)$, $1/\sqrt{(a^2 - x^2)}$, $1/\sqrt{(a^2 + x^2)}$, $1/\sqrt{(x^2 - a^2)}$.
Use of substitution for integrals involving quadratic surds.	In more complicated cases, substitutions will be given.
The derivation and use of simple reduction formulae.	Students should be able to derive formulae such as $nI_n = (n - 1)I_{n-2}$, $n \geqslant 2$, for $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, $I_{n+2} = \dfrac{2\sin(n + 1)x}{n + 1} + I_n$ for $I_n = \int \dfrac{\sin nx}{\sin x} dx$, $n > 0$.
The calculation of arc length and the area of a surface of revolution.	The equation of the curve may be given in cartesian or parametric form. Equations in polar form will not be set.

5. Vectors

What students need to learn:

The vector product $\mathbf{a} \times \mathbf{b}$ and the triple scalar product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$.	The interpretation of $\lvert\mathbf{a} \times \mathbf{b}\rvert$ as an area and $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ as a volume.
Use of vectors in problems involving points, lines and planes.	Students may be required to use equivalent cartesian forms also.
The equation of a line in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.	Applications to include (i) distance from a point to a plane, (ii) line of intersection of two planes, (iii) shortest distance between two skew lines.
The equation of a plane in the forms $\mathbf{r} \cdot \mathbf{n} = p$, $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$.	Students may be required to use equivalent cartesian forms also.

6. Further matrix algebra

What students need to learn:

Linear transformations of column vectors in two and three dimensions and their matrix representation.	Extension of work from F1 to 3 dimensions.
Combination of transformations. Products of matrices.	The transformation represented by $\mathbf{AB}$ is the transformation represented by $\mathbf{B}$ followed by the transformation represented by $\mathbf{A}$.
Transpose of a matrix.	Use of the relation $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.
Evaluation of 3×3 determinants.	Singular and non-singular matrices.
Inverse of 3×3 matrices.	Use of the relation $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
The inverse (when it exists) of a given transformation or combination of transformations.	
Eigenvalues and eigenvectors of 2×2 and 3×3 matrices.	Normalised vectors may be required.
Reduction of symmetric matrices to diagonal form.	Students should be able to find an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is diagonal.